Neutrino transport in 6D spherical coordinates using spectral methods

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Plan

1. Introduction
2. Different approximations
3. Transport equation in spherical coordinates
4. Numerical results

Plan

- Neutrino transport in 3D stellar collapse is the less well modelled mechanism, due to high number (6) of dimensions in phase space.
- It seems to be however crucial for the explosion mechanism especially for high mass progenitors.
- Our context of study applies both for photon and neutrino transport. Use of results coming for photon transfer.

Assumptions

- Zero mass approximation for neutrinos
- Only one flavor
- The polarisation of the neutrino will be neglected.
- The plasma will be supposed to be at the Local Thermal Equilibrium (L.T.E.) and at rest.
- General relativistic terms are not taken into account.

The Boltzmann equation

Let be $F(x, y, z, p_x, p_y, p_z)$ the distribution function in the phase space of an assembly of particles. The change of the number of particles in the elementary volume of the phase space $D^3\vec{x}\, D^3\vec{p} = DxDyDzDpxDpyDpz$ is given by

$$\frac{D}{dt} F(\vec{x}, \vec{p}) D^3\vec{x}\, D^3\vec{p} = CT \tag{1}$$

where D/dt is the molecular derivative or (Liouville operator) and CT is a collision term.
By expliciting the the molecular derivative D/dt we obtain

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} + \frac{dp_x}{dt} \frac{\partial}{\partial p_x} + \frac{dp_y}{dt} \frac{\partial}{\partial p_y} + \frac{dp_z}{dt} \frac{\partial}{\partial p_z} \right) F = CT \tag{2}$$

where $\vec{v}$ is the velocity of the particle. We have

$$\vec{v} = \frac{1}{m} \vec{p} \quad \text{and then } \frac{d\vec{p}}{dt} = \vec{f}$$

The case of photons and neutrinos

No forces on neutrinos:

$$\frac{1}{c}\frac{\partial F}{\partial t} + \omega^i \frac{\partial F}{\partial x^i} = CT \quad ,\omega^i = \frac{v^i}{c}, \quad \vec{\omega}\cdot\vec{\omega} = 1 \tag{3}$$

where c is the light velocity.
Once known the differential cross section $\sigma_d\left(\vec{\omega}\cdot\vec{\omega'}, \nu \to \nu'\right)$ of neutrinos and plasma particles, the neutrinos emission term $S(\nu, , \vec{x})$, and the neutrino absorption cross section $\sigma_a(\nu)$ where $\nu = E/h$ is the "frequency" of the neutrino , the collision term can be computed easily: We we can write

$$\begin{aligned} \frac{D}{dt} F d\vec{x} d\vec{p} = d\vec{x} d\vec{p} [n(\vec{x}, t)(S(\nu, \vec{x}) - \sigma_a F \\ + \int_0^\infty d\nu' \int_{4\pi} d\vec{\omega}' \sigma_d(\nu \to \nu', \vec{\omega}\cdot\vec{\omega'} c) F(\nu', \vec{\omega'})[1 \pm \frac{c^3}{2\nu^2} F(\nu, \vec{\omega}] \\ - \int_0^\infty d\nu' \int_{4\pi} d\vec{\omega'} \sigma(\nu \to \nu', \vec{\omega}\cdot\vec{\omega}') F(\nu, \vec{\omega})[1 \pm \frac{c^3}{2\nu^2} F(\nu', \vec{\omega}')]. \end{aligned} \tag{4}$$

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Coherent scattering

Assumptions:

- The plasma is at rest in the laboratory frame.
- The velocity of the scattering particles can be neglected
- The energy $h\nu$ and rest energy (of the scattering particle) mc^2 ratio between the scattered particle, (neutrinos,or photons) and the scattering target (leptons or hadrons) is very small. $(h\nu/mc^2 << 1)$

Energy of the scattered particle:

$$\sigma_d(\nu \to \nu^{'}, \vec{\omega} \cdot \vec{\omega}^{'}) = \sigma_d(\vec{\omega} \cdot \vec{\omega}^{'})\delta(\nu - \nu') \tag{5}$$

$$\sigma_t = \int_0^{\infty} d\nu^{'} \int_{4\pi} d\omega^{'} \sigma_{\mathbf{d}}(\nu \to \nu^{'} . \tilde{\omega} \cdot \tilde{\omega}^{'}) \tag{6}$$

$$\sigma_d^{Th} = 2r_e^2(1 + (\vec{\omega} \cdot \vec{\omega}^{'})^2)\, \delta(\nu - \nu^{'}), \quad \sigma_t^{Th} = \frac{8\pi}{3} r_e^2 \tag{7}$$

where $r_e = e^2/(m_e c^2)$ is the classical radius of the electron.

$$\sigma_d^n = \frac{1}{4\pi}(A + B\cos(\vec{\omega} \cdot \vec{\omega}^{'})\, \delta(\nu - \nu^{'})), \quad \sigma_t^n = A \tag{8}$$

Assumptions

- The plasma is in Local Thermal Equilibrium (LTE), its thermal energy kT to mass energy ratio (KT/mc^2) is small
- the neutrino (or photon) energy $\gamma < 1$

$$\frac{\partial F(\gamma,\vec{\omega})}{\partial t} + \vec{\omega}\cdot\nabla_{\vec{x}}F(\gamma,\vec{\omega}) =$$
$$-n(\vec{x},t)\sigma_{Th}\left[F(\gamma,\vec{\omega}) + \frac{3}{16\pi}\int_{4\pi} d\vec{\omega}^{'}\left[1+(\vec{\omega}\cdot\vec{\omega}^{'})^2\right]F(\gamma,\vec{\omega}^{'})\right]$$
$$+\frac{\partial}{\partial\gamma}\left[\left(\alpha\gamma^2\frac{\partial}{\partial_\gamma}+\gamma^2-2\alpha\gamma\right)\int_{4\pi} d\vec{\omega}^{'}F(\gamma,\vec{\omega}^{'})\right] + n(\vec{x},t)\frac{3\sigma_{Th}}{128\pi^2}\times$$
$$\int_{4\pi} d\vec{\omega}^{'}\int_{4\pi} d\omega^{``}\left[1-\vec{\omega}^{'}\cdot\vec{\omega}^{``}+(\vec{\omega}^{'}\cdot\vec{\omega}^{``})^2-(\vec{\omega}^{'}\cdot\vec{\omega}^{``})^3\right]\partial_\gamma\left(F(\gamma,\vec{\omega}^{'})F(\gamma,\vec{\omega}^{``})\right) \quad (9)$$

$\lambda_c = h/m_e c, (\sigma)_{Th} = 8\pi e^4/(3m_e^2c^4)$, $\alpha = kT/m_e c^2, \gamma = h\nu/m_e c^2$

The Fokker Planck approximation(2)

Two characteristic times:

- $\tau_{is} = 1/(n\,\sigma_{Th}\,c)$ isotropisation time
- $\tau_{Bo} = 1/(\alpha\,n\sigma_{Th}\,c)$ "bosonisation" time
- $\alpha =<< 1$ so $\tau_{Bo} >> \tau_{is}$.

$F = F(t, \gamma)$: Kompaneet equation:

$$\frac{1}{c}\frac{\partial F}{\partial t} + \frac{\partial}{\partial_\gamma}\left[\alpha\gamma^2\frac{\partial F}{\partial\gamma} + (\gamma^2 - 2\alpha\gamma)F \pm \frac{1}{2}F^2\right] = 0 \tag{10}$$

If we integrate on the energy γ both sides of the Eq.(10) we have the conservation of the number of photons

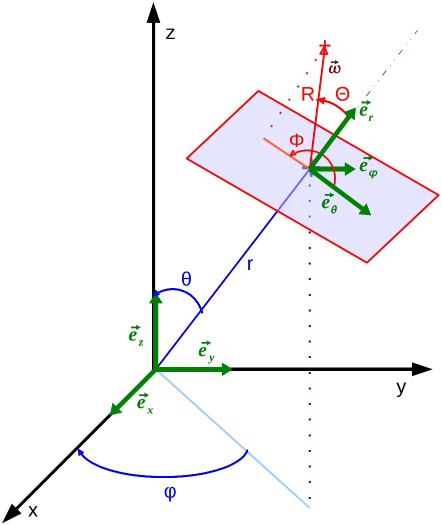
$$\frac{\partial}{\partial t}\int_0^\infty F(\gamma, t)d\gamma = 0 \tag{11}$$

The steady state solution of the Eq.(10) is the Bose distribution

$$F(\gamma) = 2\gamma^2\left(\exp\left(\frac{\gamma+\mu}{\alpha}\right) \pm 1\right)^{-1} \tag{12}$$

Plan

System of coordinates in phase space



System of coordinates: $r, \ \theta, \ \phi, \ \Theta, \ \Phi, \ \gamma$ Expression of the Liouville operator:

$$\mathcal{L}_{sph} = \cos\Theta\,\frac{\partial}{\partial r} + \frac{1}{r}(\sin\Theta\cos\Phi\,\frac{\partial}{\partial\theta} + \frac{\sin\Theta\sin\Phi}{\sin\theta}\,\frac{\partial}{\partial\phi} - \sin\Theta\,\frac{\partial}{\partial\Theta} - \sin\Theta\sin\Phi\frac{\cos\theta}{\sin\theta}\,\frac{\partial}{\partial\Phi}) \tag{13}$$

Transport equation:

$$\frac{1}{c}\frac{\partial F}{\partial t} + \mathcal{L}_{\rm sph}\,F + \sigma_t F - \int_{4\pi}\sigma_D(\vec{\omega}\cdot\vec{\omega}`)F\sin\Theta^{'}\,d\Theta^{'}\,d\Phi^{'} + D^2_\gamma F \tag{14}$$

Conservative form:

$$\frac{1}{c}\frac{\partial}{\partial t}\left(Fr^2\sin\theta\sin\Theta\right) + \frac{\partial}{\partial r}\left(Fr^2\sin\theta\sin\Theta\cos\Theta\right) + \frac{\partial}{\partial\theta}\left(Fr\sin^2\Theta\sin\theta\cos\Phi\right) - \frac{\partial}{\partial\Theta}\left(Fr\sin^2\Theta\sin\theta\right) + \frac{\partial}{\partial\phi}\left(Fr\sin^2\Theta\sin\Phi\right) - \frac{\partial}{\partial\Phi}\left(Fr\sin^2\Theta\cos\theta\sin\Phi\right) + r^2\sin\Theta\sin\theta\left(\sigma_t F - \int_{4\pi}\sigma_d(\vec{\omega}\cdot\vec{\omega}^{'})\,F\sin\Theta^{'}\,d\Theta^{'}\,d\Phi`\right) = 0 \tag{15}$$

Equation of conservation

After an integration on r, θ, ϕ, Θ, Φ We obtain

$$\frac{\partial N}{\partial t} + J_{R_2} - J_{R_1} = 0 \tag{16}$$

where

$$N = \int_{R_1}^{R_2} r^2\, dr \int_{4\pi} \sin\theta\, d\theta\, d\phi \int_{4\pi} \sin\Theta d\Phi F \tag{17}$$

is the number of photons (or neutrinos) and J_{R2}

$$J_{R_2} = R_2^2 \int_{4\pi} \sin\theta\, d\,\theta\, d\phi \int_{4\pi} \sin\Theta\cos\Theta\, d\Theta\, d\Phi F(t, R_2, \theta, \phi, \Theta, \Phi) \tag{18}$$

is the flux of photons (neutrinos) ingoing (outgoing) into the surface $r = R_2$ of the spherical shell $R_1 \leq r \leq R_2$ The same one for J_{R_1}. Induced process taken into account: the number of photons/neutrino is conserved, because σ_d satisfies the detailed balance condition

$$\sigma_d(\gamma^{`} \rightarrow \gamma, \vec{\omega}^{'} \rightarrow \vec{\omega}) = \sigma_d(\gamma \rightarrow \gamma^{'}, \vec{\omega} \rightarrow \vec{\omega}^{'}) \tag{19}$$

Expression of impulsion in spherical coordinates

Cartesian components $\omega_x, \ \omega_y, \ \omega_z$ of the vector $\vec{\omega}$ as function of $\theta, \ \phi$. Θ and Φ:

$$\omega_x = \cos\Theta\sin\theta\cos\phi + \sin\Theta\cos\Phi\cos\theta\cos\phi - \sin\Theta\sin\Phi\sin\phi \tag{20}$$

$$\omega_y = \cos\Theta\sin\theta\sin\phi + \sin\Theta\cos\Phi\cos\theta\sin\phi + \sin\Theta\sin\Phi\cos\phi \tag{21}$$

$$\omega_z = \cos\Theta\cos\theta - \sin\Theta\cos\Phi\sin\theta \tag{22}$$

The following properties hold

$$\omega_x^2 + \omega_y^2 + \omega_z^2 = 1 \tag{23}$$

$$\mathcal{L}_{sph}\,\omega_x = 0, \quad \mathcal{L}_{sph}\,\omega_y = 0 \quad \mathcal{L}_{sph}\,\omega_z = 0 \tag{24}$$

$$\mathcal{L}_{sph}\,x = \omega_x, \quad \mathcal{L}_{sph}\,y = \omega_y \quad \mathcal{L}_{sph}\,z = \omega_z \tag{25}$$

$$\int_{4\pi}\omega_x^2\mathbf{d}\omega = \int_{4\pi}\omega_y^2\mathbf{d}\omega = \int_{4\pi}\omega_z^2\mathbf{d}\omega = \frac{4\pi}{3} \tag{26}$$

Boundary conditions

2D case:

$$\frac{\partial F}{\partial t} + \cos\Theta \frac{\partial F}{\partial r} - \frac{1}{r}\sin\Theta \frac{\partial F}{\partial \Theta} = CT \tag{27}$$

- Operator is degenerate in $\Theta = \frac{\pi}{2}$: in a spherical shell, propagation operates from the inner boundary to the outer one for $\Theta \in [0, \pi/2]$; and the opposite for $\Theta \in [\pi/2, \pi]$.
- Then inner BC in r for $\Theta \in [0, \pi/2]$, and outer BC or $\Theta \in [\pi/2, \pi]$.
- Only continuity in Θ for the solution can (and must) be imposed in the equation.
- This explains why only slow convergence is attained by usual P-N (Spherical harmonic) approximations for transport equation. [Pomraning, 73]
- The cure: split the computational domain in two parts ($\Theta \in [0, \pi/2]$ and $\Theta \in [\pi/2, \pi]$) and then match...

Plan

Time evolution of an uniform distribution

Here in after, the absorption and emission terms are vanishing.
We consider a box containing an uniform plasma of density $n = n_0$ and a photon distribution $F = F(\omega_x, \omega_y, \omega_z)$ Let us consider a coherent scattering. In this case the differential operator involving the energy γ in the Eq.(9) disappears and we have to treat 5-D problem, because $\vec{\omega} = \vec{\omega}(\theta, \phi, \Theta, \Phi)$ and Liouville operator $\mathcal{L}_{sph}$ contains the operator ∂_r Taking into account the relationship given by the Eq.(24) we have

$$\mathcal{L}_{sph} F = 0 \tag{28}$$

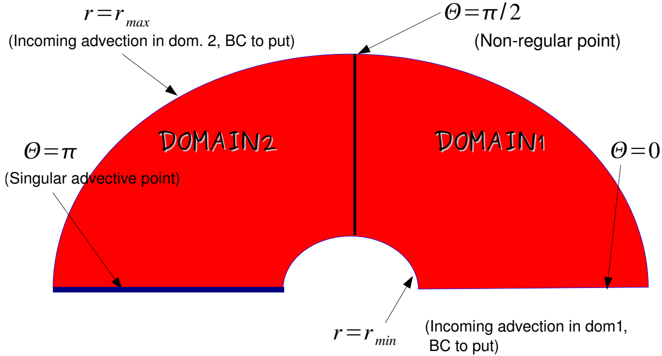
In spite the fact that $\mathcal{L}_{sph} F = 0$, the code computes the Liouville operator in order to test the stability of the algorithm and the CPU time.
The initial conditions are

$$F(\theta, \phi, \Theta, \Phi, 0) = \left(1 + 2 * \omega_x + \omega_y + \frac{1}{2}\omega_z\right)^4 \text{ for } \quad 0 \leq \Theta \leq \frac{\pi}{2} \tag{29}$$

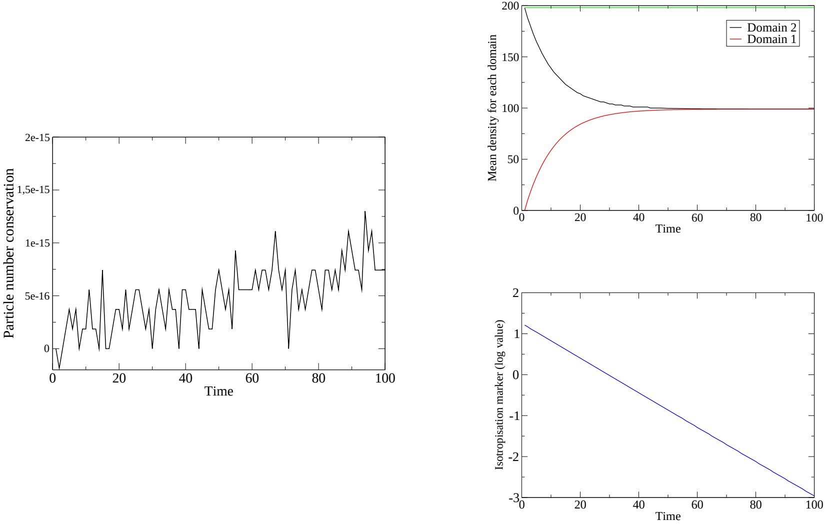
and

$$F(\theta, \phi, \Theta, \Phi) = 0 \text{ for } \quad \frac{\pi}{2} < \Theta \leq \pi \tag{30}$$

- Technical details: Results showed are obtained with a station alpha XP-1000 1Ghz of clock frequency, $1\,G_0$ RAM.
- Typical time with $N_r = 33$, $N_\theta = 17$, $N_\phi = 16$, $N_\Theta = 25$, $N_\Phi = 17$ is 50sec per time step. 75 per cent of the CPU time is spent in the routine computing the Liouville operator.
- Computational domain: Resolution on a shell $R_1 \leq r \leq R_2$, $0 \leq \Theta \leq \pi$.



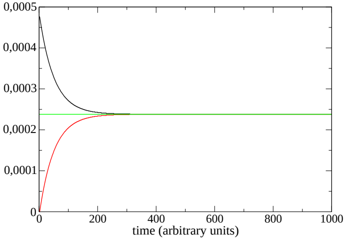
Isotropisation process for a uniform distribution

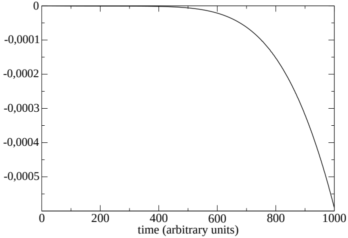


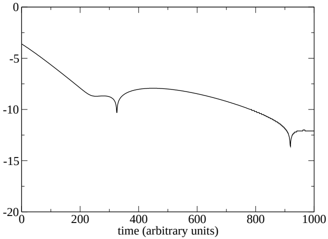
Time evolution of an uniform distribution: The non coherent case

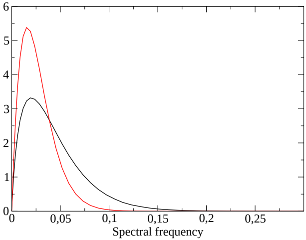
- The next example is the same one as before, but with the evolution of the energy.
- We show the results with the same distribution in the phase space, and a black body energy distribution with a temperature half of the plasma temperature $(kT/m_e c^2 = .01)$
- The final distribution of the photon is a Bose distribution (Because of the conservation of the number of photons).
- With $N_\gamma = 33$ and the same number of points for $\theta, \phi, \Theta, \Phi$ as before, the CPU time per time step is 83 sec, where half of time is spent in computing the Liouville operator.

Evolution of a uniform distribution: "Bosonisation" with Fokker-Planck equation

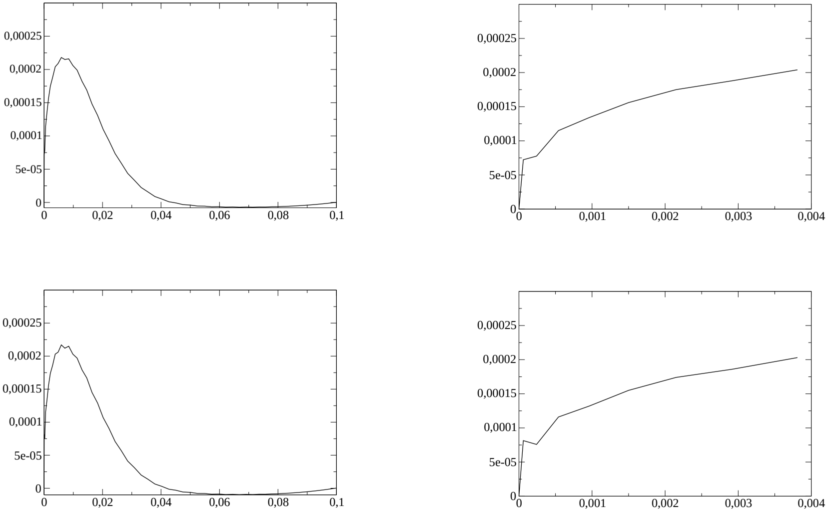




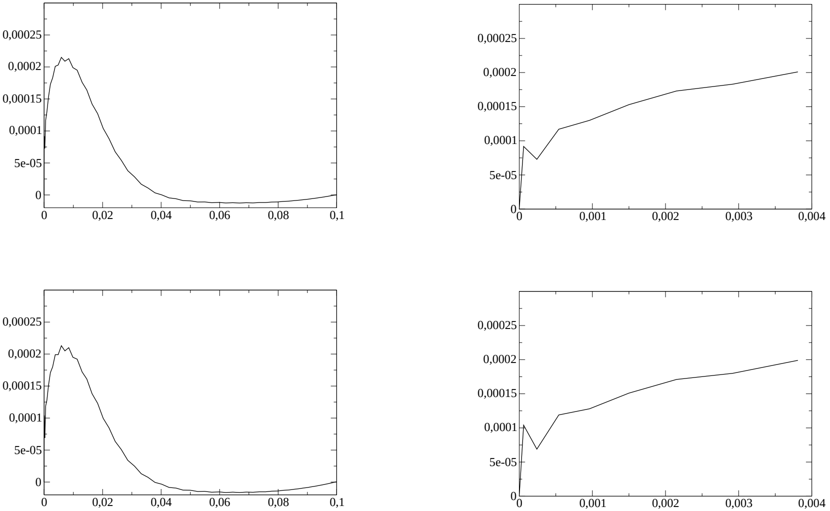




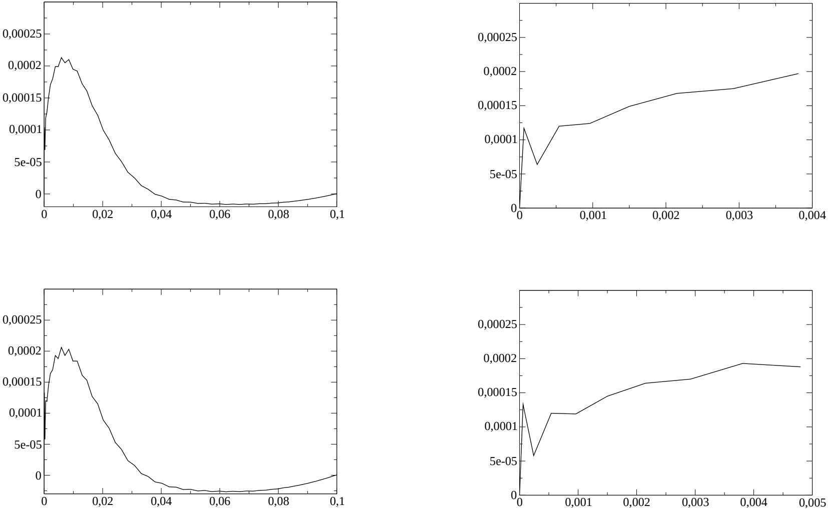
Evolution of a uniform distribution: "Bosonisation" with Fokker-Planck equation



Evolution of a uniform distribution: "Bosonisation" with Fokker-Planck equation

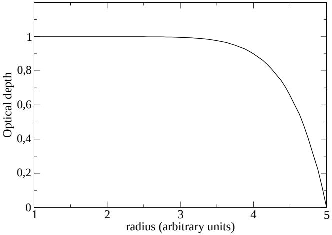


Evolution of a uniform distribution: "Bosonisation" with Fokker-Planck equation



We consider the following problem:

- Radiating black body at $r = R_1$
- Computational domain $R_1 \leq r \leq R_2$ is filled with a plasma with a density



$$n(r, \theta, \phi) = n_0[1 + .1(r \sin\theta \cos\phi \cos\theta)]\,[1 - (1 - r/R_1)/(1 - R_2/R1)]^8$$

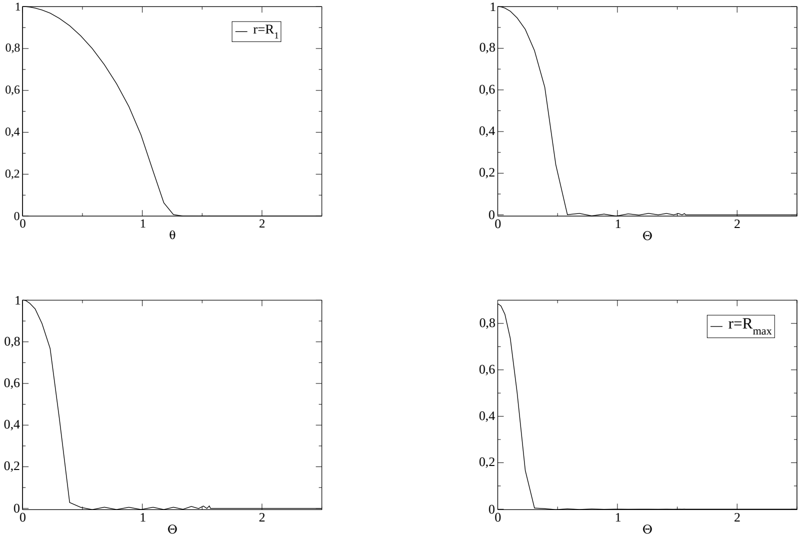
in the domain $R_1 \leq r \leq R_2$

- The initial conditions are $F = 0$ and B.C. at $r = R_1$ are (Lambert law)

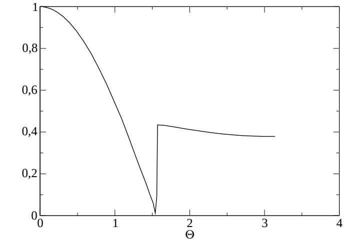
$$F(R_1, \theta, \phi, \Theta, \Phi) = F_0 \cos\Theta \tag{31}$$

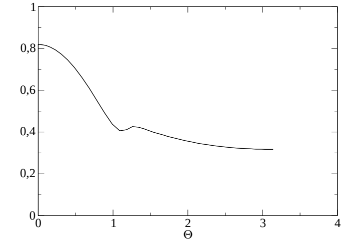
- The numerical scheme is finite difference for r. The CPU time per time step with $N_r = 300$, $N_\theta = 17$, $N_\phi = 16$ $N_\Theta = 24$, $N_\Phi = 16$ is 540 sec (wallclock time). With a spectral solution for the r dimension, it would be five times less.

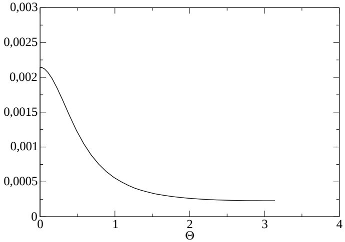
Full 5D advection,vanishing optical depth: data at different radiuses

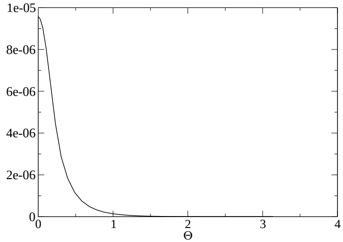


Full 5D advection, optical depth=5: data at different radiuses









Conclusions

1) We have showed the possibility to simulate the neutrinos (photons) at 5-D with one processor in the case of a plasma at rest by using a spectral algorithm. Thanks this algorithm, the singularities appearing in Liouville operator can be handled easily.

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2) The simulation in 6-D case would require parallel computation.

3) The case of a plasma not at rest can be solved by treating the problem in the plasma frame and coming back in the star frame via a Lorentz transform. No estimation of the CPU time is made today.

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4) The problem of neutrinos scattering with electrons, for which the Fokker Planck approximation is not valid, is not yet solved.

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