Modern Theory of Nuclear Forces

**Lecture 1:** Introduction & first look into ChPT
- Historical overview
- Conventional approach
  - General structure of the 2N force
  - Modern “high-precision” NN potentials
  - Beyond two nucleons
- Chiral Perturbation Theory
  - Introduction
  - Chiral symmetry of QCD
  - Effective Lagrangian

**Lecture 2:** EFTs for two nucleons

**Lecture 3:** Nuclear forces from chiral EFT
Historical overview

1935  Yukawa suggests that nucleons interact via exchange of massive scalar particles

1936-42  Extension to pseudoscalar/pseudovector exchange particles by Proca, Kemmer, Moller, Rosenfeld and Schwinger

1946  Existence of an isovector pseudoscalar meson (pion) predicted by Pauli

1947  Experimental discovery of pions by Lattes, Muirhead, Occhialini and Powell

1951  Taketani, Nakamura, Sasaki introduce new concept: long-range $(1\pi)$, medium-range $(1\pi + 2\pi)$ and core (???)

1950s  $2\pi$-exchange potential studied by Taketani et al., Brückner, Watson, ...

1960s  Discovery of heavy mesons, OBE models

70s, 80s  Dispersion and inverse scattering theory, BE and quark cluster models, phenomenology

80s, 90s  High-precision potentials ($\chi^2_{data} \sim 1$): AV18, CD Bonn, Nijm I,II, Reid 93, ...

since 91  Chiral effective field theory
2N force: general structure

Available vectors: \( \vec{r}_1, \vec{r}_2, \vec{p}_1, \vec{p}_2, \vec{\sigma}_1, \vec{\sigma}_2 \) and isovectors: \( \tau_1, \tau_2 \)

Invariance under translations and Galilei transformations: \( V_{2N}(\vec{r}_1, \vec{r}_2, \vec{p}_1, \vec{p}_2) = V_{2N}(\vec{r}', \vec{p}) \)

where \( \vec{r}' = \vec{r}_1 - \vec{r}_2, \quad \vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2) = -i\vec{\nabla}_r \)

Invariance under rotations, space reflection, time reversal & isospin rotations

\[
\begin{pmatrix}
1, & \vec{\sigma}_1 \cdot \vec{\sigma}_2, & S_{12}(\vec{r}) , & S_{12}(\vec{p}), & \vec{L} \cdot \vec{S}, & (\vec{L} \cdot \vec{S})^2 \\
\end{pmatrix}
\otimes
\begin{pmatrix}
1, & \tau_1 \cdot \tau_2 \\
\end{pmatrix}
\]

where: \( \vec{L} = \vec{r} \times \vec{p}, \quad \vec{S} = 1/2(\vec{\sigma}_1 + \vec{\sigma}_2), \quad S_{12}(\vec{x}) = 3(\vec{\sigma}_1 \cdot \vec{x})(\vec{\sigma}_2 \cdot \vec{x}) - \vec{\sigma}_1 \cdot \vec{\sigma}_2 \)

All operators are to be multiplied with scalar functions of \( r^2, p^2, \vec{r} \cdot \vec{p} \) or, equivalently, \( r^2, p^2, L^2 \) since \( (\vec{r} \cdot \vec{p})^2 = r^2p^2 - L^2 \) such that the resulting \( V \) is hermitian.

Momentum-space representation \( \langle \vec{p}' | V | \vec{p} \rangle \)

\[
\begin{pmatrix}
1, & \vec{\sigma}_1 \cdot \vec{\sigma}_2, & S_{12}(\vec{q}), & S_{12}(\vec{k}), & i\vec{S} \cdot \vec{q} \times \vec{k}, & \vec{\sigma}_1 \cdot \vec{q} \times \vec{k} \vec{\sigma}_2 \cdot \vec{q} \times \vec{k} \\
\end{pmatrix}
\otimes
\begin{pmatrix}
1, & \tau_1 \cdot \tau_2 \\
\end{pmatrix}
\]

where \( \vec{q} = \vec{p}' - \vec{p}, \quad \vec{k} = \vec{p}' + \vec{p} \).

The operators are to be multiplied with scalar functions of \( q^2, k^2, \vec{q} \cdot \vec{k} \).
Class I (isospin invariant forces): \[ [V_f^{2N}, \ T] = 0 \rightarrow V_f^{2N} = \alpha + \beta (\tau_1 \cdot \tau_2) \]

Class II (charge independence breaking):
\[ [V_f^{2N}, \ T] \neq 0, \ [V_f^{2N}, \ P_{cr}] = [V_f^{2N}, \ (T)^2] = 0 \rightarrow V_f^{2N} = \alpha \tau_1^3 \tau_2^3 \]
\[ P_{cr} = \exp(i\pi T_2) \]

Evidence: \[ 1/2(\delta_{nn}^\alpha + \delta_{pp, str}^\alpha) \neq \delta_{np}^\alpha \]
In particular: \[ a_{nn}^{1S0} \simeq -18.9 \text{ fm}, \quad a_{pp, str}^{1S0} \simeq -17.5 \text{ fm}, \quad a_{np}^{1S0} = -23.74(2) \text{ fm} \]

Class III (charge symmetry breaking, no isospin mixing):
\[ [V_{fIII}^{2N}, \ T] \neq 0, \ [V_{fIII}^{2N}, \ P_{cr}] \neq 0, \ [V_{fIII}^{2N}, \ (T)^2] = 0 \rightarrow V_{fIII}^{2N} = \alpha (\tau_1^3 + \tau_2^3) \]

Evidence: \[ \delta_{nn}^\alpha \neq \delta_{pp, str}^\alpha, \ \text{BE difference of mirror nuclei, …} \]

Class IV (charge symmetry breaking and isospin mixing):
\[ [V_{fIV}^{2N}, \ T] \neq 0, \ [V_{fIV}^{2N}, \ P_{cr}] \neq 0, \ [V_{fIV}^{2N}, \ (T)^2] \neq 0 \rightarrow V_{fIV}^{2N} = \alpha (\tau_1^3 - \tau_2^3) + \beta [\tau_1 \times \tau_2]^3 \]

Evidence: different neutron/proton analyzing powers in np scattering, …
From potential to phase shifts

Nonrelativistic Lippmann-Schwinger (LS) equation in partial waves (finite-range potential)

\[ T_{i'i}^{s'j}(p', p) = V_{i'i}^{s'j}(p', p) + \sum_l \int_0^\infty \frac{d\tilde{p} \tilde{p}^2}{(2\pi)^3} V_{i'i}^{s'j}(p', \tilde{p}) \frac{m}{p^2 - \tilde{p}^2 + i\eta} T_{i'i}^{s'j}(\tilde{p}, p) \]

where \( V_{i'i}^{s'j}(p', p) \equiv \langle p', l'sj|\hat{V}|p, lsj \rangle \) and \( T_{i'i}^{s'j}(p', p) = T_{i'i}^{s'j}(p', p, k) \bigg|_{k=p} \equiv \langle p', l'sj|\hat{T}(k)|p, lsj \rangle \bigg|_{k=p} \)

- Uncoupled: \( s = 0, 1, \ l = l' = j \), e.g. \( ^1S_0, ^1P_1, ^3P_1, \ldots \), and \( s = 1, \ l = l' = 1, \ j = 0 \) (\( ^3P_0 \))
- Coupled: \( s = 1, \ l, l' = j \pm 1 \), e.g. \( ^3S_1-^3D_1 \) \( \rightarrow \) LS equation is a 2 x 2 matrix equation

Once LS equation is solved using standard methods, phase shifts can be obtained as follows:

\[ S_{i'i}^{s'j}(k) = \delta_{i'i} - \frac{i}{8\pi^2} km T_{i'i}^{s'j}(k, k) \]

\[ S_{i'i}^{s'j}(k) = e^{2i\delta} \text{ in the uncoupled case} \]

\[ \begin{pmatrix} S_{-+}^{1-} & S_{++}^{1+} \\ S_{+-}^{1-} & S_{++}^{1+} \end{pmatrix} = \begin{pmatrix} e^{2i\delta_-} \cos 2\epsilon & ie^{i(\delta_- + \delta_+)} \sin 2\epsilon \\ ie^{i(\delta_- + \delta_+)} \sin 2\epsilon & e^{2i\delta_+} \cos 2\epsilon \end{pmatrix} \]

(Stapp parametrization in the coupled case)

Once S-matrix is known, all NN scattering observables can be calculated straightforwardly.

Electromagnetic interaction between point-like nucleons up to and including $\mathcal{O}(\alpha^2)$- and $\mathcal{O}(1/m_N^2)$-terms:

$$V_{EM}(pp) = V_{C}^{\text{improved}} + V_{VP} + V_{MM}(pp), \quad V_{EM}(np) = V_{MM}(np), \quad V_{EM}(nn) = V_{MM}(nn)$$

**Improved Coulomb potential** (leading $1/m_N^2$-corrections to $1\gamma + 2\gamma$-exchange)  
*Austin, de Swart '83*

$$V_{C}^{\text{improved}} = \frac{\alpha'}{r} \left( 1 - \frac{\alpha}{m_p r} \right) \quad \text{with} \quad \alpha' = \alpha \frac{m_p^2 + 2k^2}{m_p \sqrt{m_p^2 + k^2}}$$

**Vacuum polarization**  
*Ueling '35, Durand III '57*

$$V_{VP} = \frac{2\alpha \alpha'}{3\pi} \int_1^\infty dx e^{-2m_{ee}rx} \left( 1 + \frac{1}{2x^2} \right) \frac{(x^2 - 1)^{1/2}}{x^2} ,$$

**Magnetic moment interaction**  
*Schwinger'48; Breit '55,'62; Stoks, de Swart, PRC 42 (1990) 1235*

$$V_{MM}(pp) = -\frac{\alpha}{4m_p^2 r^3} \left[ \mu_p^2 S_{12} + (6 + 8\kappa_p) \vec{L} \cdot \vec{S} \right] ,$$

$$V_{MM}(np) = -\frac{\alpha\kappa_n}{2m_n r^3} \left[ \frac{\mu_p}{2m_p} S_{12} + \frac{1}{m} \left( \vec{L} \cdot \vec{S} + \frac{1}{2} \vec{L} \cdot (\vec{\sigma}_1 - \vec{\sigma}_2) \right) \right] ,$$

$$V_{MM}(nn) = -\frac{\alpha\mu_n^2}{4m_n^2 r^3} S_{12}$$

---

![Graph showing the effect of the MM interaction](image)
**Strategy:** take into account the known longest-range physics due to EM force and $1\pi$-exchange

\[
V_{1\pi}(\vec{q}) \propto \frac{\vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q}}{q^2 + M_{\pi}^2} \vec{\tau}_1 \cdot \vec{\tau}_2 \quad \text{or in r-space:} \quad V_{1\pi}^{\text{long}}(\vec{r}) \propto e^{-M_{\pi}r} \left[ S_{12} \left(1 + \frac{3}{M_{\pi}r} + \frac{3}{(M_{\pi}r)^2}\right) + \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right] \vec{\tau}_1 \cdot \vec{\tau}_2
\]

and parametrize the medium- and short-range contributions in a most general way.

**Example: AV18 potential**

*Wiringa, Stoks, Schiavilla '94*

- Local r-space potential
- EM contributions multiplied by short-range functions to account the finite size of the nucleons
- Regularized OPEP including isospin breaking due to $M_{\pi^\pm} \neq M_{\pi^0}$
- Some additional phenomenological shorter range isospin-breaking terms
- Medium-range ($r \sim (2M_{\pi})^{-1}$) contributions of Yukawa-type, short-range ones of the Woods-Saxon type
- 40 adjustable parameters fitted to 4301 pp and np scattering data, $\chi^2_{\text{datum}} = 1.09$
Other phenomenological potentials

- OBE motivated nonlocal (Nijm I, 41 parameters; CD Bonn, 43 parameters) and local (Nijm II, 47 parameters; Reid93, 50 parameters) potentials. All have $\chi^2_{\text{datum}} \sim 1$ and are define in the partial wave basis.

- BE models (Nijm93, Bonn): less parameters but higher $\chi^2_{\text{datum}}$

- INOY, CD Bonn + Δ, inverse scattering, $V_{\text{low-k}}$, …

phaselke.png

phase shifts generated by http://nn-online.org/
"...replacement of field interactions by two-body action-at-a-distance potentials is a poor approximation in nuclear physics."

Lippmann-Schwinger equation: \[ \mathcal{T} = \mathcal{V} + \mathcal{V} \mathcal{T} \]

Set of three coupled integral equations:
\[ U_1 = \mathcal{T} + \mathcal{T} U_2 + \mathcal{T} U_3 \]
\[ Faddeev '65; Alt, Grassberger, Sandhas '67, '72 \]

3N calculations based on phenomenological NN potentials show evidence for missing 3N forces (e.g. the underbinding of \( ^3\)H by about 1 MeV).
Most general parametrization of the 3NF seems not feasible:

- too many possible structures (> 100)
- too scarce data base available
- too involved calculations

need guidance from theory

Three-nucleon force models
Fujita-Miyazawa, Brazil, Tucson-Melbourne, Urbana IX, Illinois, …

The strategy:
Take various combinations of 2N and 3N potentials, adjust parameters of the 3NF model to reproduce e.g. $^3$H BE and apply resulting $V_{2N} + V_{3N}$ to scatt. observables.
Successes and failures

Elastic scattering observables

Deuteron breakup

Inclusion of the 3NF sometimes leads to improvements, sometimes — not. Situation, in part, chaotic.

Need a **theoretical** approach which would:
- be based on QCD,
- yield consistent many-body forces,
- be systematically improvable,
- allow for error estimation

\[ \text{chiral effective field theory} \]
Further reading

Some modern “high-precision” nucleon-nucleon potentials

- **Wiringa, Stoks, Schiavilla, Phys. Rec. C51 (95) 38** [Argonne V18]
- **Machleidt, Phys. Rev. C63 (01) 024001** [CD Bonn 2000]
- **Machleidt, Slaus, J. Phys. G27 (01) R69** [review article]

Three-nucleon force models

- **Fujita, Miyazawa, Prog. Theor. Phys. 17 (57) 360** [Fujita-Miyazawa 3NF model]
- **Coon, Han, Few-Body Syst. 30 (01) 131** [Tucson-Melbourne 3NF model]
- **Coelho, Das, Robilotta, Phys. Rev. C28 (83) 1812** [Brazilian 3NF model]
- **Pudliner, Pandharipande, Carlson, Pieper, Wiringa, Phys. Rev. C56 (97) 1720** [Urbana IX 3NF model]
- **Pieper, Wiringa, Ann. Rev. Nucl. Part. Sci. 51 (01) 53** [Illinois 3NF model]

Review articles on 3N scattering & 3N force effects

- **Glöckle, Witala, Huber, Kamada, Golak, Phys. Rept. 274 (96) 107**
- **Kalantar-Nayestanaki, E.E., Nucl. Phys. News 17 (07) 22**
QCD: nuclear force is due to residual color force

However...

- non-perturbative at low energy
- “wrong” degrees of freedom

Nonperturbative methods

- lattice QCD
  Nemura, Ishii, Aoki, Detmold, ...
- effective field theory
  Weinberg, ...
- large-$N_c$ expansion
  Kaplan, Savage, Cohen, ...
An effective (field) theory is an approximate theory whose scope is to describe phenomena which occur at a chosen length (or energy) range.

Example: multipole expansion for electric potentials

\[
V \propto \int \frac{\rho(\vec{r})}{d} d^3 r
\]

\begin{align*}
&= \int \frac{\rho(\vec{r})}{\sqrt{R^2 + 2rR \cos \theta + r^2}} d^3 r \\
&= \sum_{n=0}^{\infty} \frac{1}{R^{n+1}} \int r^n P_n(\cos \theta) \rho(\vec{r}) d^3 r \\
&= q \frac{1}{R} + P \frac{1}{R^2} + Q \frac{1}{R^3} + \ldots
\end{align*}

the sum converges rapidly for \( a \ll R \)
“if one writes down the most general possible Lagrangian, including all terms consistent with the assumed symmetry principles, and then calculates $S$-matrix elements with this Lagrangian to any order in perturbation theory, the result will simply be the most general possible $S$-matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles”

S. Weinberg, Physica A96 (79) 327

- identify the symmetries of the underlying theory,
- construct the most general $\mathcal{L}_{\text{eff}}$ in terms of relevant d.o.f. and consistent with the symmetries,
- do standard quantum field theory with the effective Lagrangian.
Chiral symmetry of QCD Lagrangian

\[ \mathcal{L}_{\text{QCD}} = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \bar{q}(i \slashed{D} - \mathcal{M}) q = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \bar{q}_L i \slashed{D} q_L + \bar{q}_R i \slashed{D} q_R - \bar{q}_L \mathcal{M} q_R - \bar{q}_R \mathcal{M} q_L \]

\( SU(2)_L \times SU(2)_R \) invariant breaks chiral symmetry

Left- and right-handed quark fields: \( q_{L,R} = \frac{1}{2} (1 \pm \gamma_5) q \).

Chiral group is a group of independent rotations of \( q_{L,R} \) in the flavor space.

For 2 flavors: \( G = SU(2)_L \times SU(2)_R \) and \( \begin{cases} \frac{q_L}{G} & \rightarrow & \frac{q'_L}{g_L} = g_L q_L \\ \frac{q_R}{g_R} & \rightarrow & \frac{q'_R}{g_R} = g_R q_R \end{cases} \) with \( g_{L,R} \in SU(2)_{L,R} \)

Chiral \( SU(2) \) Lie algebra:

\[
[\Gamma^L_i, \Gamma^L_j] = i \epsilon_{ijk} \Gamma^L_k \\
[\Gamma^R_i, \Gamma^R_j] = i \epsilon_{ijk} \Gamma^R_k \\
[\Gamma^L_i, \Gamma^R_j] = 0
\]

or

\[
[V_i, V_j] = i \epsilon_{ijk} V_k \\
[A_i, A_j] = i \epsilon_{ijk} A_k
\]

with \( V_i = \Gamma^R_i + \Gamma^L_i \), \( A_i = \Gamma^R_i - \Gamma^L_i \)

vector (isospin) generators

axial generators

\( m_u, d \) small \( \Rightarrow \mathcal{L}_{\text{QCD}} \) is approximately \( M_{\pi}^2/M_{\rho}^2 \sim 0.03 \) chiral invariant
There is strong evidence that chiral symmetry of QCD is spontaneously broken down to the isospin group:

- Only isospin but not chiral multiplets are observed in the particle spectrum (axial charges would lead to parity doublets).

- Triplet of unnaturally light pseudoscalar mesons (pions) — natural candidates for Goldstone bosons.

- Scalar quark condensate:
  \[ \langle 0 | \bar{q}q | 0 \rangle_{\overline{M}_S, 2 \text{GeV}} = -(273 \pm 12 \text{ MeV})^3 \]
  (Lattice QCDSF/UKQCD, Schierholz et al. ‘07)

- Further theoretical arguments:
  Vafa & Witten ‘84; ‘t Hooft ‘80; Coleman & Witten ‘80
asymptotically observed states as effective DOF → EFT

spontaneously broken approximate $\chi$-symmetry of QCD plays a crucial role

light ($M_\pi$) and heavy ($M_\rho$) scales well separated

Cannot derive $\mathcal{L}_{\text{eff}}$ write most general expression consistent with $\chi$-symmetry, i.e.:
- include all possible $\chi$-invariant terms,
- include all terms that break $\chi$-symmetry in the same way as $\bar{q}mq$ in $\mathcal{L}_{\text{QCD}}$ does.

Consider the pure Goldstone boson sector in the chiral limit.
- How to write down most general $\chi$-invariant $\mathcal{L}_{\text{eff}}$?
- How do $\pi$’s transform under $G$?
- Isospin subgroup $H \subseteq G$ realized linearly ($\pi$’s build an isospin triplet).
- Chiral group necessarily realized nonlinearly ($\text{SU}(2)_L \times \text{SU}(2)_R$ isomorphic to $\text{SO}(4)$ need at least 4 dimensions to construct a nontrivial linear realization)
Chiral rotations & pion fields

Infinitesimal $SO(4)$ rotation of the 4-vector $(\pi_1, \pi_2, \pi_3, \sigma)$:

$$
\begin{pmatrix}
\bar{\pi}_1 \\
\bar{\pi}_2 \\
\bar{\pi}_3 \\
\sigma
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\bar{\pi}'_1 \\
\bar{\pi}'_2 \\
\bar{\pi}'_3 \\
\sigma'
\end{pmatrix}
= 1 + \bar{\theta}^V \cdot \bar{V} + \bar{\theta}^A \cdot \bar{A}
\begin{pmatrix}
\bar{\pi}_1 \\
\bar{\pi}_2 \\
\bar{\pi}_3 \\
\sigma
\end{pmatrix}
$$

where: $\bar{\theta}^V \cdot \bar{V} = \begin{pmatrix}
0 & -\theta^V_3 & \theta^V_2 & 0 \\
\theta^V_3 & 0 & -\theta^V_1 & 0 \\
-\theta^V_2 & \theta^V_1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$ and $\bar{\theta}^A \cdot \bar{A} = \begin{pmatrix}
0 & 0 & 0 & \theta^A_1 \\
0 & 0 & 0 & \theta^A_2 \\
-\theta^A_1 & -\theta^A_2 & -\theta^A_3 & 0
\end{pmatrix}$

One reads off: $\bar{\pi}' = \bar{\pi} + \bar{\theta}^V \times \bar{\pi} + \bar{\theta}^A \sigma$ and $\sigma' = \sigma - \bar{\theta}^A \cdot \bar{\pi}$

Switch to the nonlinear realization of $SO(4)$:
only 3 out of 4 components of the vector $(\bar{\pi}, \sigma)$ are independent, i.e. $\bar{\pi}^2 + \sigma^2 = F^2$

$$
\sigma = \sqrt{F^2 - \bar{\pi}^2}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\bar{\pi} \rightarrow \bar{\pi}' = \bar{\pi} + \bar{\theta}^V \times \bar{\pi} \\
\bar{\pi} \rightarrow \bar{\pi}' = \bar{\pi} + \bar{\theta}^A \sqrt{F^2 - \bar{\pi}^2}
\end{pmatrix}
\begin{array}{c}
\text{linear under } \bar{\theta}^V \\
\text{nonlinear under } \bar{\theta}^A
\end{array}
$$

It is more convenient to use a $2 \times 2$ matrix notation:

$$
U = \frac{1}{F} \left( \sigma I + i \bar{\pi} \cdot \bar{\tau} \right)
\xrightarrow{\text{nonlinear realization}}
U = \frac{1}{F} \left( I \sqrt{1 - \bar{\pi}^2} + i \bar{\pi} \cdot \bar{\tau} \right)
$$

Chiral rotations: $U \rightarrow U' = LUR^\dagger$ with $L = e^{-i/2(\bar{\theta}^V - \bar{\theta}^A) \cdot \bar{\pi}}$ and $R = e^{-i/2(\bar{\theta}^V + \bar{\theta}^A) \cdot \bar{\pi}}$
The above realization of $G$ is not unique. How does this non-uniqueness affect $S$-matrix?

- All realizations of $G$ are equivalent to each other by means of nonlinear field redefinitions $\tilde{\pi} \to \tilde{\pi}' = \tilde{\pi} F[\tilde{\pi}], \quad F[0] = 1$ \textit{(Coleman, Callan, Wess & Zumino '69)}
- Such field redefinitions do not affect $S$-matrix \textit{(Haag '58)}

**Derivative expansion for the effective Lagrangian** $\mathcal{L}_{\text{eff}} = \mathcal{L}^{(2)}_{\pi} + \mathcal{L}^{(4)}_{\pi} + \ldots$

- 0 derivatives: $UU\dagger = U\dagger U = 1$ — plays no role
- 2 derivatives: $\text{Tr}(\partial_{\mu}U \partial^{\mu}U\dagger) \xrightarrow{\text{gC}} \text{Tr}(L\partial_{\mu}UR\dagger R \partial^{\mu}U\dagger L\dagger) = \text{Tr}(\partial_{\mu}U \partial^{\mu}U\dagger)$

  $$\mathcal{L}^{(2)}_{\pi} = \frac{F^2}{4} \text{Tr}(\partial_{\mu}U \partial^{\mu}U\dagger)$$

- 4 derivatives: $[\text{Tr}(\partial_{\mu}U \partial^{\mu}U\dagger)]^2$, $\text{Tr}(\partial_{\mu}U \partial_{\nu}U\dagger) \text{Tr}(\partial^{\mu}U \partial^{\nu}U\dagger)$, $\text{Tr}(\partial_{\mu}U \partial^{\nu}U\dagger \partial_{\rho}U \partial^{\nu}U\dagger)$

  (terms with $\partial_{\mu}\partial_{\nu}U$, $\partial_{\mu}\partial_{\nu}\partial_{\rho}U$, $\partial_{\mu}\partial_{\nu}\partial_{\rho}\partial_{\sigma}U$ can be eliminated via EOM/partial integration)

  

  \ldots

What is the meaning of $F$?

Axial current from $\mathcal{L}^{(2)}_{\pi}$: $J_{A \mu}^i = i \text{Tr}[\tau^i(U\dagger \partial_{\mu}U - U \partial_{\mu}U\dagger)] = -F \partial_{\mu}\pi^i + \ldots$

$$\langle 0 | J_{A \mu}^i | \pi^j(\vec{p}) \rangle \equiv ip_{\mu} F_\pi \delta^{ij} \quad \Rightarrow \quad F = F_\pi = 92.4 \text{ MeV}$$
How to account for explicit $\chi$-symmetry breaking due to nonvanishing quark masses?

Trick (method of external sources): $\delta \mathcal{L}_{\text{QCD}} = -\bar{q}\mathcal{M}q \bigg|_{\mathcal{M}=m}$

$$-\bar{q}\mathcal{M}q = -\bar{q}_L\mathcal{M}q_R - \bar{q}_R\mathcal{M}q_L$$

is $\chi$-invariant if: $\mathcal{M} \xrightarrow{G} \mathcal{M}' = g_R \mathcal{M}g_L^{-1} = g_L \mathcal{M}g_R^{-1}$

write down all possible $\chi$-invariant terms with $\mathcal{M}$ and then set $\mathcal{M} = m$

The leading (i.e. no $\partial_\mu$ and $\propto \mathcal{M}$) SB term in $\mathcal{L}_{\text{eff}}$:

$$\mathcal{L}_{\text{SB}} = \frac{BF^2}{2} \text{Tr}[(U + U^\dagger)\mathcal{M}] \bigg|_{\mathcal{M}=m} = 2BF^2 m_q - B m_q \bar{\pi}^2 + \mathcal{O}(\bar{\pi}^4) \quad \Rightarrow \quad M_\pi^2 = 2m_q B + \mathcal{O}(m_q^2)$$

The LEC $B$ is related to the scalar quark condensate via $\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = -BF^2 + \mathcal{O}(\mathcal{M})$

Notice: the generalized scenario (Stern et al. '91) in which $2m_q B \ll M_\pi^2$ is ruled out by recent data on $\pi\pi$ scatt. length.
Effective Lagrangian

\[ \mathcal{L}^{(2)} = \frac{F^2}{4} \left[ \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \text{Tr}(U \chi + U^\dagger \chi) \right], \]

\[ \mathcal{L}^{(4)} = L_1[\text{Tr}(\partial_\mu U^\dagger \partial^\mu U)]^2 + L_2 \text{Tr}(\partial_\mu U^\dagger \partial_\nu U) \text{Tr}(\partial^\mu U^\dagger \partial^\nu U) + L_3 \text{Tr}(\partial_\mu U^\dagger \partial^\mu U \partial_\nu U^\dagger \partial^\nu U) \]
\[ + L_4 \text{Tr}(\partial_\mu U^\dagger \partial^\mu U) \text{Tr}(U \chi + U^\dagger \chi) + L_5 \text{Tr}(\partial_\mu U^\dagger \partial^\mu U (U \chi + U^\dagger \chi)) + L_6[\text{Tr}(U \chi + U^\dagger \chi)]^2 \]
\[ + L_7[\text{Tr}(U \chi - U^\dagger \chi)]^2 + L_8 \text{Tr}(\chi U \chi U + \chi U^\dagger \chi U^\dagger) \]

where \( \chi = 2BM \).

- Only those terms are shown which do not involve external sources (there are 3 more terms which describe the interaction of GBs with external fields).

- The Lagrangian is shown for the SU(3) x SU(3) case. Some terms are redundant in the case of SU(2) x SU(2) chiral symmetry.

- How to calculate observables? 

  → see next lecture…