Development of collective behavior in nuclei

- Results primarily from correlations among valence nucleons.

- Instead of pure “shell model” configurations, the wave functions are mixed – linear combinations of many components.

- Leads to a lowering of the collective states and to enhanced transition rates as characteristic signatures.
Coherence and Transition Rates

Consider simple case of $N$ degenerate levels: $2^+$

$\Delta E = (N - 1)V$

$\Psi = a\phi_1 + a\phi_2 + \ldots + a\phi_N$

where $a = \frac{1}{\sqrt{N}}$

$$\left( \sum_{i=1}^{N} a^2 = \frac{N}{N} = 1 \right)$$

Consider transition rate from $2_1^+ \rightarrow 0_1^+$

$$B(E2; 2_1^+ \rightarrow 0_1^+) = \frac{1}{2J_i + 1} \left( 0_1^+ \| E2 \| 2_1^+ \right)^2$$

$$\left( 0_1^+ \| E2 \| 2_1^+ \right) = \left( 0_1^+ \| E2 \| \psi \right) = a \sum_{i=1}^{N} \left( 0_1^+ \| E2 \| \phi_i \right)$$

The more configurations that mix, the stronger the $B(E2)$ value and the lower the energy of the collective state.

Fundamental property of collective states.
Low Lying \[\rightarrow\] \textbf{Quadrupole Vibrations}

Angular Momentum \(2^+\)

\[
\begin{align*}
E &= 2E_{ph} & |2\rangle & & 4^- & 2^- & 0^- & \text{Superpose} & 2 \\
\text{Quad vibs} & & & & & & & & \\
E &= E_{ph} & |1\rangle & 2^+ & \text{Quad. vib. rel. to} \\
& & & & & & & & \text{g.s.} \\
E &= 0 & |0\rangle & 0^+ \\
\end{align*}
\]
Phonon creation and destruction operators

Quadrupole case

\[ b^\dagger \ , \ b^\dagger_{2\mu} \quad \text{(drop "2\mu")} \]

\[ |n_b\rangle = \begin{cases} \sqrt{n_b} |n_b - 1\rangle & n_b \geq 1 \\ 1 & n_b = 0 \end{cases} \]

\[ b^\dagger \ |n_b\rangle = \sqrt{n_b + 1} |n_b + 1\rangle \]

\[ b^\dagger \ |0\rangle = 0 \]

\[ b^\dagger |0\rangle = |n_b = 1\rangle = \varphi_{1\text{ phonon}} \]

\[ b^\dagger b = \text{number operator—counts } n_b \]

\[ b^\dagger b |n_b\rangle = b^\dagger \sqrt{n_b} |n_b - 1\rangle = \sqrt{n_b} \sqrt{(n_b - 1) + 1} |n_b\rangle \]

\[ b^\dagger b |n_b\rangle = n_b |n_b\rangle \]
Electromagnetic Transitions in the phonon model

$0^+, 2^+, 4^+ \quad 2 \text{ ph}$

$2^+ \quad 1 \text{ ph}$

$0^+$

$E2$ operator is proportional to the annihilation operator, $b$, for a phonon.

$$
\langle n_f | b | n_i \rangle = \sqrt{n_i} \langle n_f | n_i - 1 \rangle
= \sqrt{n_i} \delta_{n_f, n_i - 1}
$$

a) $E2$ transition probability

$$
[ \propto \langle | | \rangle^2 ] \propto n_i
$$

b) Selection rule $\Delta n = 1$

c) Branching ratio

$$
\frac{B(E2; n = 2 \rightarrow n = 1)}{B(E2; n = 1 \rightarrow n = 0)} = 2
$$

d) $B(E2; n = 2 \rightarrow 0^+ \text{ g.s.}) = 0$ --- forbidden
$B(E2)$ VALUES FOR DECAY OF
MULTI-PHONON STATES
<table>
<thead>
<tr>
<th>Ion</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4^+$</td>
<td>2.51</td>
</tr>
<tr>
<td>$0^+$</td>
<td>2.29</td>
</tr>
<tr>
<td>$2^+$</td>
<td>2.16</td>
</tr>
<tr>
<td>$2^-$</td>
<td>1.33</td>
</tr>
<tr>
<td>$2^+$</td>
<td>0.99</td>
</tr>
<tr>
<td>$0^+$</td>
<td>0.56</td>
</tr>
<tr>
<td>$2^+$</td>
<td>0.51</td>
</tr>
<tr>
<td>$0^-$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{Ni}^{60}$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{Zn}^{64}$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{Se}^{76}$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{Pd}^{106}$</td>
<td>0</td>
</tr>
<tr>
<td>$4^+$</td>
<td>2.28</td>
</tr>
<tr>
<td>$0^+$</td>
<td>2.05</td>
</tr>
<tr>
<td>$2^+$</td>
<td>2.04</td>
</tr>
<tr>
<td>$4^+$</td>
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<td>$2^+$</td>
<td>1.21</td>
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<tr>
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<td>1.13</td>
</tr>
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<td>$2^-$</td>
<td>1.23</td>
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<tr>
<td>$2^+$</td>
<td>0.56</td>
</tr>
<tr>
<td>$0^+$</td>
<td>0.56</td>
</tr>
<tr>
<td>$0^+$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{Cd}^{114}$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{Sn}^{118}$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{Te}^{122}$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{Ba}^{134}$</td>
<td>0</td>
</tr>
</tbody>
</table>

$V \sim C_2\beta^2$
Octupole Vibrations

\[ 3^- \]

2-phonon \[ 3^- \otimes 3^- \Rightarrow J = 0^+, 2^+, 4^+, 6^+ \]

A few examples beginning to be known

\[ ^{96}_{40} \text{Zr}, \quad ^{146}_{64} \text{Gd} \]

Multi-phonon \quad Octupole – Quadrupole

\[ 3^- \otimes 2^+ \]
Deformed, ellipsoidal, rotational nuclei

Let's look at a typical example and see the various aspects of structure it shows

Axially symmetric case
Axial asymmetry
Rotational states built on (superposed on) vibrational modes

\[ V \sim C_2\beta^2 + C_3 \beta^3 \cos 3\gamma + C_4\beta^4 \]

\[ \hbar^2 / 2l = 11.1 \text{ keV} \]

Ground or equilibrium state

Vibrational excitations

Rotational states

- \(8^+\)
- \(6^+\)
- \(4^+\)
- \(2^+\)
- \(0^+\)
Axial asymmetry (Triaxiality)
(Specified in terms of the coordinate $\gamma$ (in degrees), either from 0 → 60 or from -30 → +30 degrees – zero degrees is axially symmetric)

$$V(\gamma)$$

$\gamma$ - rigid

$\gamma$ - soft (flat, unstable)

$$V \sim C_2 \beta^2 + C_3 \cos 3\gamma \beta^3 + C_4 \beta^4$$

Note: for axially symm. deformed nuclei, MUST have a large $C_3$ term
Axial Asymmetry in Nuclei – two types

\[ E \sim \Lambda (\Lambda + 3) \sim J (J + 6) \]

Note staggering in gamma band energies

Gamma Rigid

Gamma Soft

Wilets-Jean, Gamma unstable
Use staggering in gamma band energies as signature for the kind of axial asymmetry
Overview of yrast energies

Can express energies as $E \sim J (J + X)$
Now that we know some simple models of atomic nuclei, how do we know where each of these structures will appear? How does structure vary with Z and N? What do we know?

• Near closed shells nuclei are spherical and can be described in terms of a few shell model configurations.

• As valence nucleons are added, configuration mixing, collectivity and, eventually, deformation develop. Nuclei near mid-shell are collective and deformed.

• The driver of this evolution is a competition between the pairing force and the p-n interaction, both primarily acting on the valence nucleons.
Estimating the properties of nuclei

We know that $^{134}$Te (52, 82) is spherical and non-collective.

We know that $^{170}$Dy (66, 104) is doubly mid-shell and very collective.

What about:

$^{156}$Te (52, 104) $^{156}$Gd (64, 92) $^{184}$Pt (78, 106) ???

All have 24 valence nucleons. What are their relative structures ???
Valence Proton-Neutron Interaction

Development of configuration mixing, collectivity and deformation – competition with pairing

Changes in single particle energies and magic numbers

Partial history: Goldhaber and de Shalit (1953); Talmi (1962); Federman and Pittel (late 1970’s); Casten et al (1981); Heyde et al (1980’s); Nazarewicz, Dobacewski et al (1980’s); Otsuka et al (2000’s); Cakirli et al (2000’s); and many others.
The idea of “both” types of nucleons – the p-n interaction

Sn – Magic: no valence p-n interactions

Both valence protons and neutrons
If p-n interactions drive configuration mixing, collectivity and deformation, perhaps they can be exploited to understand the evolution of structure.

Let's assume, just to play with an idea, that all p-n interactions have the same strength. This is not realistic since the interaction strength depends on the orbits the particles occupy, but, maybe, on average, it might be OK.

How many valence p-n interactions are there? $N_p \times N_n$

If all are equal then the integrated p-n strength should scale with $N_p \times N_n$

*The $N_pN_n$ Scheme*
Valence Proton-Neutron Interactions

Correlations, collectivity, deformation. Sensitive to magic numbers.

\[ \frac{N_p N_n}{(N_p + N_n)} \]

\( P = \frac{N_p N_n}{(N_p + N_n)} \)

n-p interactions per pairing interaction
The $N_pN_n$ scheme: Interpolation vs. Extrapolation
Predicting new nuclei with the $N_p N_n$ Scheme

All the nuclei marked with x’s can be predicted by INTERpolation
Competition between pairing and the p-n interactions

A simple microscopic guide to the evolution of structure

(The next slides allow you to estimate the structure of any nucleus by multiplying and dividing two numbers each less than 30)

(or, if you prefer, you can get the same result from 10 hours of supercomputer time)
Valence p-n interaction: Can we measure it?

\[
\delta_{Vpn}(Z, N) = \frac{1}{4} \left( B(ZN) - B(Z)B(N) + B(ZN^2) - B(Z^2N) + B(Z^2N^2) - B(ZN^2) \right)
\]

Int. of last two n with Z protons, N-2 neutrons and with each other

Int. of last two n with Z-2 protons, N-2 neutrons and with each other

Empirical average interaction of last two neutrons with last two protons
Empirical interactions of the last proton with the last neutron

\[ \delta V_{pn}(Z, N) = -\frac{1}{4}\{[B(Z, N) - B(Z, N - 2)] - [B(Z - 2, N) - B(Z - 2, N - 2)]\} \]
p-n / pairing

\[ P = \frac{N_p N_n}{N_p + N_n} \sim \frac{p-n}{\text{pairing}} \]

Pairing int. \( \sim 1 \) MeV, \( p-n \sim 200 \) keV

Hence takes \( \sim 5 \) p-n int. to compete with one pairing int.

\[ P \sim 5 \]
Comparison with the data
The IBA

The Interacting Boson Approximation Model
A very simple phenomenological model, that can be extremely parameter-efficient, for collective structure

• Why the IBA
• Basic ideas about the IBA, including a primer on its Group Theory basis
• The Dynamical Symmetries of the IBA
• Practical calculations with the IBA
Drastic simplification of shell model

- Valence nucleons
- Only certain configurations
- Simple Hamiltonian – interactions

“Boson” model because it treats nucleons in pairs
2 fermions → boson
Why do we need to simplify – why not just calculate with the Shell Model????

The Need for Simplification in Multiparticle Spectra

Example: How many 2+ states?

\[ \text{# nucl.} \]

\[
\begin{array}{ccc}
2 & d_{5/2}^2 & 1 \\
4 & d_{5/2} \cdot g_{7/2} & \geq 7
\end{array}
\]

\[
\begin{align*}
|d_{5/2}^2 J = 2, g_{7/2}^2 J = 0 \rangle, & |d_{5/2}^2 J = 0, g_{7/2}^2 J = 2 \rangle \\
|d_{5/2}^2 J = 4, g_{7/2}^2 J = 2 ; J = 2 \rangle, & |d_{5/2}^2 J = 2, g_{7/2}^2 J = 4 ; J = 2 \rangle \\
|d_{5/2}^2 J = 4, g_{7/2}^2 J = 6 ; J = 2 \rangle, & |d_{5/2}^2 g_{7/2} J = 1, d_{5/2} g_{7/2} J = 1 ; J = 2 \rangle, \\
|d_{5/2}^2 J = 4, g_{7/2}^2 J = 4 ; J = 2 \rangle.
\end{align*}
\]

\[ ^{154}\text{Sm}_{82} \]

cl. sh. \[ 50 \quad 82 \]

\[ N_p = 12 \quad N_n = 10 \]

12 val. \( \pi \) in 50 – 82

10 val. \( \nu \) in 82 – 126

\[ \begin{array}{cc}
82 & 126 \\
81/2 & p_{1/2} \\
d_{5/2} & p_{3/2} \\
h_{11/2} & f_{5/2} \\
d_{5/2} & f_{7/2} \\
g_{7/2} & h_{9/2}
\end{array} \]

How many 2+ states subject to Pauli Principle limits?

\[ 3 \times 10^{14} ! ! ! \]

\[ ^{154}\text{Sm} \quad 2^+ \text{ states within the valence shell space} \]
Shell Model Configurations

Fermion configurations

Roughly, gazillions!! Need to simplify

The IBA
Boson configurations (by considering only configurations of pairs of fermions with $J = 0$ or $2$.)
Assume *valence* fermions couple in pairs to bosons of spins 0+ and 2+

<table>
<thead>
<tr>
<th>0+</th>
<th>s-boson</th>
</tr>
</thead>
<tbody>
<tr>
<td>2+</td>
<td>d-boson</td>
</tr>
</tbody>
</table>

s boson is like a Cooper pair
d boson is like a generalized pair

- Valence nucleons only
- \( s, d \) bosons – creation and destruction operators

\[
H = H_s + H_d + H_{\text{interactions}}
\]

Number of bosons fixed: \( N = n_s + n_d \)

\( = \frac{1}{2} \) # of val. protons + \( \frac{1}{2} \) # val. neutrons
Why $s$, $d$ bosons?

- Lowest state of all $e-e$ nuclei is $0^+$
- $\delta$ - fct gives $0^+$ ground state
- First excited state in non-magic $e-e$ nuclei almost always $2^+$
- $\delta$ - fct gives $2^+$ next above $0^+$
Modeling a Nucleus

Why the IBA is the best thing since baseball, a jacket potato, aceto balsamico, Mt. Blanc, raclette, pfannekuchen, baklava, ….

\[ ^{154}\text{Sm} \longrightarrow \text{Shell model} \longrightarrow 3 \times 10^{14} \text{ 2}^+ \text{ states} \]

Need to truncate

IBA assumptions

1. Only valence nucleons
2. Fermions \( \rightarrow \) bosons
   - \( J = 0 \) (s bosons)
   - \( J = 2 \) (d bosons)

Is it conceivable that these 26 basis states are correctly chosen to account for the properties of the low-lying collective states?

IBA: \( 26 \text{ 2}^+ \text{ states} \)
Why the IBA ????

• Why a model with such a drastic simplification – Oversimplification ???

• Answer: Because it works !!!!!

• By far the most successful general nuclear collective model for nuclei

• Extremely parameter-economic
Note key point:

- Bosons in IBA are pairs of fermions in valence shell

Number of bosons for a given nucleus is a fixed number

\[ ^{154}_{62} \text{Sm}_{92} \quad N_\pi = 6 \quad 5 = N_\nu \quad \Rightarrow \quad N_B = 11 \]

Basically the IBA is a Hamiltonian written in terms of s and d bosons and their interactions. It is written in terms of boson creation and destruction operators.
Where the IBA fits in the pantheon of nuclear models

- Shell Model \(\text{Sph. Def.}\) - (Microscopic)
- Geometric – (Macroscopic)
- Third approach — “Algebraic”

Dynamical Symmetries

Group Theoretical

Phonon-like model with microscopic basis explicit from the start.
**IBA** has a deep relation to Group theory

That relation is based on the operators that create, destroy $s$ and $d$ bosons

\[ s^+, s, \left\{ d^+, d \right\} \text{ operators} \]

\[ \text{Ang. Mom. 2} \]

\[ d^+_{\mu}, d_{\mu} \quad \mu = 2, 1, 0, -1, -2 \]

Hamiltonian is written in terms of $s, d$ operators

Since boson number is **conserved** for a given nucleus, $H$ can only contain “bilinear” terms: 36 of them.

\[ s^+ s, s^+ d, d^+ s, d^+ d \]

Gr. Theor. classification of Hamiltonian

Group is called \[ \text{U(6)} \]
Brief, simple, trip into the Group Theory of the IBA

DON’T BE SCARED

You do not need to understand all the details but try to get the idea of the relation of groups to degeneracies of levels and quantum numbers

A more intuitive name for this application of Group Theory is “Spectrum Generating Algebras”
**Review of phonon creation and destruction operators**

\[ \mathbf{b} | n_b \rangle = \sqrt{n_b} | n_b - 1 \rangle \]

\[ \mathbf{b}^\dagger | n_b \rangle = \sqrt{(n_b + 1)} | n_b + 1 \rangle \]

What is a creation operator? Why useful?

A) Bookkeeping – makes calculations very simple.

B) “Ignorance operator”: We don’t know the structure of a phonon but, for many predictions, we don’t need to know its microscopic basis.

\[ \mathbf{b}^\dagger \mathbf{b} | n_b \rangle = \mathbf{b}^\dagger \sqrt{n_b} | n_b - 1 \rangle = \sqrt{n_b} \sqrt{(n_b - 1) + 1}| n_b \rangle = n_b | n_b \rangle \]

\[ \mathbf{b}^\dagger \mathbf{b} \] is a \( \mathbf{b} \)-phonon number operator.

For the IBA a boson is the same as a phonon – think of it as a collective excitation with ang. mom. 0 (s) or 2 (d).
Concepts of group theory
First, some fancy words with simple meanings: Generators, Casimirs, Representations, conserved quantum numbers, degeneracy splitting

Generators of a group: Set of operators \( O_i \) that close on commutation.

\[
[ O_i, O_j ] = O_i O_j - O_j O_i = O_k \quad i.e., \quad \text{their commutator gives back 0 or a member of the set}
\]

For IBA, the 36 operators \( s^\dagger s, d^\dagger s, s^\dagger d, d^\dagger d \) are generators of the group U(6).

\[
ex: \quad [ d^\dagger s, s^\dagger s ] | n_d n_s \rangle = ( d^\dagger s s^\dagger s - s^\dagger s d^\dagger s ) | n_d n_s \rangle \\
= d^\dagger s n_s | n_d n_s \rangle - s^\dagger s d^\dagger s | n_d n_s \rangle \\
= (n_s - s^\dagger s) d^\dagger s | n_d n_s \rangle
\]

\[
e.g.: \quad \left[ N, s^\dagger \left( n_d - 1 \right) s \right] \Psi \\
= \sqrt{n_d + 1} \sqrt{n_s} \left[ s^\dagger \left( n_s^\dagger - n_s \frac{\Psi}{n_s - 1} \right) | n_d n_s \rangle \right] - N s^\dagger d^\dagger N \Psi \\
= d^\dagger s | n_d n_s \rangle
\]

or:

\[
[d^\dagger s, s^\dagger s] = d^\dagger s
\]
Sub-groups:

Subsets of generators that commute among themselves.

\textit{e.g.:} \( d^\dagger d \) \hspace{1cm} 25 generators—span U(5)

They conserve \( n_d \) (# \( d \) bosons)

Set of states with same \( n_d \) are the representations of the group [ U(5) ]
Simple example of dynamical symmetries, group chain, degeneracies

\[ [H, J^2] = [H, J_Z] = 0 \]

\[ J, M \] constants of motion
Let’s illustrate group chains and degeneracy-breaking.

Consider a Hamiltonian that is a function ONLY of: \( s^+s + d^+d \)

That is: \[ H = \alpha(s^+s + d^+d) = \alpha (n_s + n_d) = \alpha N \]
Now, add a term to this Hamiltonian:

\[ H' = H + b_d \]

Now the energies depend not only on \( N \) but also on \( n_d \).

States of a given \( n_d \) are now degenerate. They are "representations" of the group \( U(5) \). States with different \( n_d \) are not degenerate.
\[ H' = aN + b \]
OK, here’s the key point:

Concept of a Dynamical Symmetry

Spectrum generating algebra!!
Next time

Classifying Structure -- The Symmetry Triangle